

PERTURBATION OF A NONAUTONOMOUS PROBLEM IN \mathbb{R}^n

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ABSTRACT. In this paper we prove a stability result about the asymptotic dynamics of a perturbed nonautonomous evolution equation in \mathbb{R}^n governed by a maximal monotone operator.

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1. INTRODUCTION

The investigation of the asymptotic behavior of nonlinear equations with some dissipation property and subjected to perturbations on parameters has been matter of extensive studies of several different frameworks. The goal is to understand how the variation of parameters in the models can determine the evolution of their states. For a recent works in this topic we refer the reader to [1, 2, 3, 4, 5] and references given there.

In this paper we analyze, from the point of view of pullback attractors theory [6, 7], the asymptotic behavior of the nonlinear nonautonomous problem

$$(1.1) \quad \begin{aligned} u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) + a_\epsilon(x)|u|^{p-2}u &= B(t, u) \quad \text{in } \mathbb{R}^n \\ u(\tau) &= u_\tau \in L^2(\mathbb{R}^n), \end{aligned}$$

with $2 < p < n$, and we also derive some stability properties with respect to small variations of the functions $a_\epsilon \in C(\mathbb{R}^n, \mathbb{R})$. We assume that $a_\epsilon(x) \geq 1$, $\epsilon \in [0, 1]$, and $\|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{\epsilon \rightarrow 0} 0$. In addition, we require that a_0 satisfy

$$(1.2) \quad \int_{\mathbb{R}^n} \frac{1}{a_0(x)^{\frac{2}{p-2}}} dx < \infty.$$

Notice that a_ϵ also satisfies (1.2) for $\epsilon \in [0, \epsilon_0]$, for some $\epsilon_0 > 0$.

Concerning to the nonlinearity, we suppose that $B : \mathbb{R} \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ satisfies both conditions below:

(i) there exists a constant $L > 0$ such that

$$\|B(t, u_1) - B(t, u_2)\|_{L^2(\mathbb{R}^n)} \leq L\|u_1 - u_2\|_{L^2(\mathbb{R}^n)}, \quad \forall t \in \mathbb{R}, \forall u_1, u_2 \in L^2(\mathbb{R}^n);$$

(ii) the map $L_1 : \mathbb{R} \ni t \mapsto \|B(t, 0)\|_{L^2(\mathbb{R}^n)} \in \mathbb{R}$ is nondecreasing, absolutely continuous and bounded on compact subsets of \mathbb{R} .

Under these assumptions we state the main result of this paper:

Theorem 1.1. *For each value of the parameter $\epsilon \in [0, \epsilon_0]$, the equation (1.1) generates a nonlinear compact process, $\{S_\epsilon(t, \tau) : t \geq \tau \in \mathbb{R}\}$, in the space $L^2(\mathbb{R}^n)$, which has a family of pullback attractors $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$. Moreover, this family of pullback attractors is upper-semicontinuous in $\epsilon = 0$.*

2. FUNCTIONAL FRAMEWORK

One of the difficulties on the analysis of the asymptotic behavior of PDE's in unbounded domains, is the lack of compactness of embeddings of some functional spaces. To overcome this obstacle, some authors have been introduced weighted Sobolev spaces, see for instance [8]. In this work will use the family of auxiliary spaces

$$E_\epsilon = \{u \in W^{1,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a_\epsilon |u|^p dx < \infty\}.$$

Next lemma is a parameter dependent adaptation of a similar result in [8, 9] which we prove here for reader's convenience.

Lemma 2.1. *The space E_ϵ endowed with norm $\|u\|_{E_\epsilon} = \left[\int_{\mathbb{R}^n} (|\nabla u|^p + a_\epsilon |u|^p) dx \right]^{\frac{1}{p}}$ is a reflexive Banach space. Furthermore $E_\epsilon \xhookrightarrow{d} L^r(\mathbb{R}^n)$, $2 \leq r \leq p^* := \frac{pn}{n-p}$, and $E_\epsilon \subset\subset L^r(\mathbb{R}^n)$, $2 \leq r < p^*$, with all embedding constants independent of $\epsilon \in [0, \epsilon_0]$.*

Proof. Notice that E_ϵ is a reflexive Banach space by Eberlein-Smulian theorem.

Now if $\theta = \frac{p}{2}$, then $\frac{\theta'}{\theta} = \frac{2}{p-2}$, where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Let be $u \in E_\epsilon$, by Hölder's Inequality,

$$\begin{aligned} \|u\|_{L^2(\mathbb{R}^n)}^2 &\leq \left[\int_{\mathbb{R}^n} \frac{1}{a_\epsilon(x)^{\frac{2}{p-2}}} dx \right]^{\frac{1}{\theta'}} \left[\int_{\mathbb{R}^n} a_\epsilon(x) |u(x)|^p dx \right]^{\frac{2}{p}} \\ &\leq \left[\int_{\mathbb{R}^n} \frac{1}{a_0(x)^{\frac{2}{p-2}}} dx + 1 \right]^{\frac{1}{\theta'}} \left[\int_{\mathbb{R}^n} a_\epsilon(x) |u(x)|^p dx \right]^{\frac{2}{p}} \leq c \|u\|_{E_\epsilon}^2. \end{aligned}$$

Furthermore, recalling that $a_\epsilon(x) \geq 1$, we also have that $E_\epsilon \hookrightarrow W^{1,p}(\mathbb{R}^n)$ with embedding constant independent of ϵ . Therefore the embedding (part of the statement) follows from Sobolev's embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$, $p \leq r \leq p^*$, and interpolation's inequality.

In order to prove the compactness part let us to consider a sequence $u_k \rightarrow 0$ in E_ϵ . Since $W^{1,p}(B(0, R)) \subset\subset L^2(B(0, R))$, for any $R > 0$, taking subsequences if necessary, we can assume that $u_k \rightarrow 0$ in $L^2(B(0, R))$. By (1.2), for each $\delta > 0$, there exists $R = R(\delta) > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(0, R)} \frac{1}{a_\epsilon(x)^{\frac{2}{p-2}}} dx < \delta,$$

and we have

$$\|u_k\|_{L^2(\mathbb{R}^n \setminus B(0, R))}^2 \leq \left[\int_{\mathbb{R}^n \setminus B(0, R)} \frac{1}{a_\epsilon(x)^{\frac{2}{p-2}}} dx \right]^{\frac{1}{\theta'}} \left[\int_{\mathbb{R}^n \setminus B(0, R)} a_\epsilon(x) |u_k(x)|^p dx \right]^{\frac{2}{p}} \leq \delta^{\frac{1}{\theta'}} \limsup_{k \rightarrow \infty} \|u_k\|_{E_\epsilon}^2.$$

Therefore

$$\|u_k\|_{L^2(\mathbb{R}^n)} = \|u_k\|_{L^2(B(0, R))} + \|u_k\|_{L^2(\mathbb{R}^n \setminus B(0, R))} \xrightarrow{k \rightarrow \infty} 0.$$

To conclude, we recall that $\{u_k\} \subset E_\epsilon \hookrightarrow L^{p^*}(\mathbb{R}^n)$ is a bounded sequence, thus for all $2 < r < p^*$, one has by interpolation's inequality that

$$\|u_k\|_{L^r(\mathbb{R}^n)} \leq \|u_k\|_{L^2(\mathbb{R}^n)}^\alpha \|u_k\|_{L^{p^*}(\mathbb{R}^n)}^{1-\alpha} \rightarrow 0.$$

□

2.1. Monotone operator. In order to rewrite the problem (1.1) in an abstract setting we consider the (nonlinear) operator $A_\epsilon : E_\epsilon \rightarrow E_\epsilon^*$ defined by

$$\langle A_\epsilon u, v \rangle_{E_\epsilon^*, E_\epsilon} = \int_{\mathbb{R}^n} (|\nabla u|^{p-2} \nabla u \cdot \nabla v + a_\epsilon(x) |u|^{p-2} uv) dx, \quad \forall v \in E_\epsilon,$$

where $\langle \cdot, \cdot \rangle_{E_\epsilon^*, E_\epsilon}$ denotes the pair of duality between E_ϵ^* and E_ϵ .

By Tartar's inequality, [10], one can show that the operators A_ϵ , $\epsilon \in [0, \epsilon_0]$, are monotone, hemicontinuous and coercive. Let us now to consider the L^r -realization, $2 \leq r \leq p^*$, of the operator A_ϵ , denoted by $A_{\epsilon,r}$, given by

$$\begin{aligned} D(A_{\epsilon,r}) &= \{u \in E_\epsilon : A_\epsilon u \in L^r(\mathbb{R}^n)\}, \\ A_{\epsilon,r}u &= A_\epsilon u, \quad \forall u \in D(A_{\epsilon,r}) \end{aligned}$$

The operators $A_{\epsilon,r}$ can also be seen as the subdifferential of the lower semicontinuous convex functions $\varphi_{\epsilon,r} : L^r(\mathbb{R}^n) \rightarrow (-\infty, \infty]$ defined by

$$(2.1) \quad \varphi_{\epsilon,r}(u) = \begin{cases} \frac{1}{p} \|u\|_{E_\epsilon}^p, & \text{if } u \in E_\epsilon \\ \infty, & \text{otherwise.} \end{cases}$$

For our purposes it is of special interest the case $r = 2$, and for shorten notation, we drop the index r and we write A_ϵ for this realization. In this setting the problem (1.1) can be written as a quasi-linear evolution equation

$$(2.2) \quad \begin{aligned} u_t^\epsilon + A_\epsilon u^\epsilon &= B(t, u^\epsilon), \\ u^\epsilon(\tau) &= u_\tau^\epsilon \in L^2(\mathbb{R}^n) \end{aligned}$$

Proposition 2.1 ([11], Proposition 3.13). *Under hypothesis (i) and (ii) on B , for all $u_\tau^\epsilon \in L^2(\mathbb{R}^n)$ there exist a unique solution $u^\epsilon = u^\epsilon(\cdot, u_\tau^\epsilon) \in W^{1,1}(\tau, \infty; L^2(\mathbb{R}^n))$ of (2.2).*

3. UNIFORM DISSIPATIVENESS

In this section we establish some uniform bounds on solutions of the problem (2.2) in order to derive existence of pullback attractors.

Lemma 3.1. *Let $u^\epsilon(\cdot, u_\tau^\epsilon) \in C([\tau, \infty), L^2(\mathbb{R}^n))$ be the global solution of (2.2). Then there exist constant T_1 (not dependent on ϵ) and a nondecreasing function $\beta_1 : \mathbb{R} \rightarrow \mathbb{R}$, such that*

$$\|u^\epsilon(t, u_\tau^\epsilon)\|_{L^2(\mathbb{R}^n)} \leq \beta_1(t), \quad \forall t \geq T_1 + \tau.$$

Proof. Multiplying (2.2) by u^ϵ and integrating over \mathbb{R}^n we have that

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + \langle A_\epsilon u^\epsilon, u^\epsilon \rangle = \langle B(t, u^\epsilon) - B(t, 0), u^\epsilon \rangle + \langle B(t, 0), u^\epsilon \rangle,$$

where $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\mathbb{R}^n)$. Thus

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + c \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^p \leq L \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + L_1(t) \|u^\epsilon\|_{L^2(\mathbb{R}^n)},$$

where $c > 0$ is the constant given in Lemma 2.1.

Taking $\theta = \frac{p}{2}$, it follows from Young's inequality that for all $\eta > 0$,

$$\frac{1}{2} \frac{d}{dt} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + c \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^p \leq \frac{1}{\theta'} \left(\frac{L}{\eta} \right)^{\theta'} + \frac{\eta^\theta}{\theta} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^p + \frac{1}{p'} \left(\frac{L_1(t)}{\eta} \right)^{p'} + \frac{\eta^p}{p} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^p.$$

Choosing $\eta > 0$ such that $\gamma = c - \left(\frac{\eta^\theta}{\theta} + \frac{\eta^p}{p} \right) > 0$, it follows from [12, Lemma 5.1] that

$$(3.2) \quad \frac{1}{2} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \left(\frac{\delta(t)}{\gamma} \right)^{\frac{2}{p}} + \left[\frac{\gamma}{2} (p-2)(t-\tau) \right]^{\frac{-2}{p-2}},$$

where $\delta(t) = \frac{1}{\theta'} \left(\frac{L}{\eta} \right)^{\theta'} + \frac{1}{p'} \left(\frac{L_1(t)}{\eta} \right)^{p'}$. Taking $T_1 > 0$ satisfying $\left[\frac{\gamma}{2} (p-2)T_1 \right]^{\frac{-2}{p-2}} \leq 1$, we have for $t \geq T_1 + \tau$ that

$$\frac{1}{2} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \left(\frac{\delta(t)}{\gamma} \right)^{\frac{2}{p}} + 1 := \beta_1(t).$$

□

Lemma 3.2. *Let $u^\epsilon(\cdot, u_\tau^\epsilon) \in C([\tau, \infty), L^2(\mathbb{R}^n))$ be the global solution of (2.2). Then there exist constant T_2 (not dependent on ϵ) and a nondecreasing function $\beta_2 : \mathbb{R} \rightarrow \mathbb{R}$, such that*

$$\|u^\epsilon(t, u_\tau^\epsilon)\|_{E_\epsilon} \leq \beta_2(t), \quad \forall t \geq T_2 + \tau.$$

Proof. By multiplying (2.2) by u_t^ϵ we have for Young's inequality that

$$\frac{1}{2} \|u_t^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{p} \frac{d}{dt} \|u^\epsilon\|_{E_\epsilon}^p \leq \frac{1}{2} (L \|u^\epsilon\|_{L^2(\mathbb{R}^n)} + L_1(t))^2,$$

and consequently, for $\theta = \frac{p}{2}$, we obtain

$$(3.3) \quad \frac{1}{p} \frac{d}{dt} \|u^\epsilon\|_{E_\epsilon}^p \leq L^2 \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + L_1(t)^2 \leq \frac{1}{\theta'} L^{2\theta'} + \frac{1}{\theta} \|u^\epsilon\|_{E_\epsilon}^p + L_1(t)^2.$$

Fix $R > 0$ and consider the real functions $a_1 = a_1(t)$ and $a_2 = a_2(t)$ given by: $a_1 := \int_t^{t+R} \frac{p}{\theta} ds = \frac{Rp}{\theta}$

and $a_2 := \frac{Rp}{\theta'} L^{2\theta'} + RpL_1(t+R)^2 \geq \int_t^{t+R} \left(\frac{p}{\theta'} L^{2\theta'} + pL_1(s)^2 \right) ds$.

Recalling (3.1), we have by integrating from t to $t+R$ that

$$\int_t^{t+R} \|u^\epsilon\|_{E_\epsilon}^p \leq \frac{1}{2} \|u^\epsilon\|_{L^2(\mathbb{R}^n)}^2 + L \int_t^{t+R} \|u^\epsilon(s)\|_{L^2(\mathbb{R}^n)}^2 ds + L_1(t+R) \int_t^{t+R} \|u^\epsilon(s)\|_{L^2(\mathbb{R}^n)} ds.$$

It follows from Lemma 3.1, for $t \geq T_1 + \tau$

$$\int_t^{t+R} \|u^\epsilon\|_{E_\epsilon}^p \leq \frac{1}{2} \beta_1(t) + RL\beta_1(t+R) + RL_1(t+R)\beta_1(t+R)^{\frac{1}{2}} := a_3(t).$$

By [12, Lemma 1.1], we obtain

$$(3.4) \quad \|u^\epsilon(t+R)\|_{E_\epsilon}^p \leq \left(\frac{a_3(t)}{R} + a_2(t) \right) e^{a_1} := \beta_2(t), \quad t \geq \tau.$$

Choosing $R = T_1$ we have for $t - \tau \geq T_2 := 2T_1$ that

$$\|u^\epsilon(t)\|_{E_\epsilon}^p \leq \beta_2(t).$$

□

4. EXISTENCE OF PULLBACK ATTRACTORS

In this section we get existence of a family $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$ of pullback attractors for the problem (2.2) as well its upper-semicontinuity in $\epsilon = 0$.

We start remembering the definition of Hausdorff semi-distance between two subsets A and B of a metric space (X, d) :

$$\text{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

Definition 4.1. *Let $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a metric space X . Given \mathcal{A} and B subsets of X , we say that \mathcal{A} pullback attracts B at time t if*

$$\lim_{\tau \rightarrow -\infty} \text{dist}_H(S(t, \tau)B, \mathcal{A}) = 0.$$

Definition 4.2. *We say that a family of subsets $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ of X is invariant relatively to the evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if $S(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, for any $t \geq \tau$.*

Definition 4.3. *A family of subsets $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ of X is called a pullback attractor for the evolution process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ if it is invariant, $\mathcal{A}(t)$ is compact for all $t \in \mathbb{R}$, and pullback attracts bounded subsets of X at time t , for each $t \in \mathbb{R}$.*

The next result guarantees the existence of pullback attractors.

Theorem 4.1 ([4]). *Let $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be an evolution process in a complete metric space X . The statements are equivalents:*

- (i) *There exist a family of compact subsets of X , $\{K(t)\}_{t \in \mathbb{R}}$, that pullback attracts bounded sets of X at time t ;*
- (ii) *The process $\{S(t, \tau) : t \geq \tau \in \mathbb{R}\}$ has a pullback attractor.*

Corollary 4.1. *For each value of the parameter $\epsilon \in [0, \epsilon_0]$, the equation (1.1) generates a nonlinear compact process, $\{S_\epsilon(t, \tau) : t \geq \tau \in \mathbb{R}\}$, in the space $L^2(\mathbb{R}^n)$, which has a family of pullback attractors $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$.*

Proof. For each value of the parameter $\epsilon \in [0, \epsilon_0]$, Proposition 2.1 guarantees that (2.2) generates a (non-linear) evolution process, $\{S_\epsilon(t, \tau) : t \geq \tau \in \mathbb{R}\}$, in the space $L^2(\mathbb{R}^n)$, defined by $S_\epsilon(t, \tau)u_\tau^\epsilon = u^\epsilon(t, u_\tau^\epsilon)$.

Lemma 3.2 shows that the family of compact sets $K_\epsilon(t) = \overline{B^{E_\epsilon}(0, \beta_2(t))}^{L^2(\mathbb{R}^n)}$ pullback attracts bounded sets of $L^2(\mathbb{R}^n)$ at time t . Thus by Theorem 4.1 there exists a family $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$ of pullback attractors for $\{S_\epsilon(t, \tau) : t \geq \tau \in \mathbb{R}\}$. \square

4.1. Upper-semicontinuity of pullback attractors. Now we prove that the family of pullback attractors $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$ is upper-semicontinuous in $\epsilon = 0$, ie, we prove that

$$\lim_{\epsilon \rightarrow 0} \text{dist}_H(\mathcal{A}_\epsilon(t), \mathcal{A}_0(t)) = 0.$$

First, let us to consider $w^\epsilon(\cdot) = u^\epsilon(\cdot, u_\tau^\epsilon) - u^0(\cdot, u_\tau^0)$. Thus $w_t^\epsilon + A_\epsilon u^\epsilon - A_0 u^0 = B(t, u^\epsilon) - B(t, u^0)$. Since $a_\epsilon \geq 1$, it follows from Tartar's inequality the existence of $\alpha > 0$ such that

$$\begin{aligned} \langle A_\epsilon u^\epsilon - A_0 u^0, w^\epsilon \rangle &= \langle |\nabla u^\epsilon|^{p-2} \nabla u^\epsilon - |\nabla u^0|^{p-2} \nabla u^0, w^\epsilon \rangle + \langle a_\epsilon |u^\epsilon|^{p-2} u^\epsilon - a_0 |u^0|^{p-2} u^0, w^\epsilon \rangle \\ &= \langle |\nabla u^\epsilon|^{p-2} \nabla u^\epsilon - |\nabla u^0|^{p-2} \nabla u^0, w^\epsilon \rangle + \langle a_\epsilon (|u^\epsilon|^{p-2} u^\epsilon - |u^0|^{p-2} u^0) + (a_\epsilon - a_0) |u^0|^{p-2} u^0, w^\epsilon \rangle \\ &\geq \alpha (\|\nabla w^\epsilon\|_{L^2(\mathbb{R}^n)}^p + \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^p) + \int_{\mathbb{R}^n} (a_\epsilon - a_0) |u^0|^{p-2} u^0 w^\epsilon dx. \end{aligned}$$

Therefore by Hölder's inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 &\leq - \int_{\mathbb{R}^n} (a_\epsilon - a_0) |u^0|^{p-2} u^0 w^\epsilon dx + \|B(t, u^\epsilon) - B(t, u^0)\|_{L^2(\mathbb{R}^n)} \|w^\epsilon\|_{L^2(\mathbb{R}^n)} \\ &\leq \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} (|u^0|^p + |u^0|^{p-1} |u^\epsilon|) dx + L \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} \left(\|u^0\|_{L^p(\mathbb{R}^n)}^p + \|u^0\|_{L^p(\mathbb{R}^n)}^{p-1} \|u^\epsilon\|_{L^p(\mathbb{R}^n)} \right) + L \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

The uniform estimates given in Lemma 3.2 lead to

$$\frac{1}{2} \frac{d}{dt} \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq M \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} + L \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2,$$

in compact subsets of \mathbb{R} . Integrating this last inequality from τ to t , we obtain

$$\|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \|u_\tau^\epsilon - u_\tau^0\|_{L^2(\mathbb{R}^n)}^2 + 2M(t - \tau) \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} + 2L \int_\tau^t \|w^\epsilon(s)\|_{L^2(\mathbb{R}^n)}^2 ds.$$

Hence, by Gronwall's Inequality

$$(4.1) \quad \|w^\epsilon\|_{L^2(\mathbb{R}^n)}^2 \leq \tilde{M} \left(\|u_\tau^\epsilon - u_\tau^0\|_{L^2(\mathbb{R}^n)}^2 + \|a_\epsilon - a_0\|_{L^\infty(\mathbb{R}^n)} \right),$$

in compact subsets of \mathbb{R} .

We can derive from this discussion the following Lemma

Lemma 4.1. *Let $\{S_\epsilon(t, \tau) : t \geq \tau \in \mathbb{R}\}$ be the process generated by the problem (2.2). If $u_\tau^\epsilon \xrightarrow{\epsilon \rightarrow 0} u_\tau^0$ in $L^2(\mathbb{R}^n)$ then $S_\epsilon(t, \tau)u_\tau^\epsilon \xrightarrow{\epsilon \rightarrow 0} S_0(t, \tau)u_\tau^0$ in $L^2(\mathbb{R}^n)$ uniformly for t in compact subsets of \mathbb{R} .*

Corollary 4.2. *The family of pullback attractors $\{\mathcal{A}_\epsilon(t) : t \in \mathbb{R}\}$ is upper-semicontinuous in $\epsilon = 0$.*

Proof. Given $\delta > 0$ let $\tau \in \mathbb{R}$ be such that $\text{dist}(S_0(t, \tau)B, \mathcal{A}_0(t)) < \frac{\delta}{2}$, where $B \supset \bigcup_{s \in \mathbb{R}} \mathcal{A}_\epsilon(s)$ is a bounded set (whose existence is guaranteed by Lemma 3.2).

Now for (4.1), there exists $\epsilon_0 > 0$ such that

$$\sup_{\xi_\epsilon \in \mathcal{A}_\epsilon(t)} \|S_\epsilon(t, \tau)\xi_\epsilon - S_0(t, \tau)\xi_\epsilon\| < \frac{\delta}{2},$$

for all $\epsilon < \epsilon_0$. Then

$$\begin{aligned} \text{dist}(\mathcal{A}_\epsilon(t), \mathcal{A}_0(t)) &\leq \text{dist}(S_\epsilon(t, \tau)\mathcal{A}_\epsilon(\tau), S_0(t, \tau)\mathcal{A}_\epsilon(\tau)) + \text{dist}(S_0(t, \tau)\mathcal{A}_\epsilon(\tau), S_0(t, \tau)\mathcal{A}_0(\tau)) \\ &= \sup_{\xi_\epsilon \in \mathcal{A}_\epsilon(\tau)} \text{dist}(S_\epsilon(t, \tau)\xi_\epsilon, S_0(t, \tau)\xi_\epsilon) + \text{dist}(S_0(t, \tau)\mathcal{A}_\epsilon(t), \mathcal{A}_0(t)) < \frac{\delta}{2} + \frac{\delta}{2}, \end{aligned}$$

which proves the upper-semicontinuity of the family of attractors. \square

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